# Quantum Feedback Control for Deterministic Entangled Photon Generation

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We present quantum feedback control for deterministic entanglement generation at the single-photon level. The protocol of controlling both total photon number and phase difference is based on the cascade structure of cavities placed in an optical closed loop, quantum nondemolition measurement with cross-Kerr interactions, and Lyapunov stability for feedback design.

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The technological feasibility of quantum communication and computation is dominated by the need to provide nonclassical states with arbitrarily high probability. Entanglement generation is of particular interest as a resource for quantum information technologies. This potential application has spurred the development of devices that can produce entanglement on demand based on two trapped ions [1], tight spatial confinement of the photons with strong atom-field coupling in a cavity [2], and ultraslow light propagation [3].

While these methods rely on highly controlled interactions, there is another possibility to produce entanglement deterministically based on conditional measurement. The conditional state preparation is substantially equivalent to a projection operation on an appropriately prepared state. Because of the stochastic nature of quantum systems, the realization of projections is probabilistic. If repeated or continuous measurement is used, the statistical property can be modified by applying a Hamiltonian depending on measurement outcomes. This scheme has been applied to a large spin system [4], in which the system is linearized due to the size of the spin so that the spin operators can be approximated to optical quadrature operators. This linearization permits the introduction of a Gaussian approximation and simplifies the analysis and design of the spin system. In general, however, it is difficult to deal with nonlinear stochastic systems such as small spins or a low number of photons because the classical Gaussian approximation is not applicable.

We describe a deterministic scheme of entanglement generation at the single-photon level between spatially separate cavities using quantum nondemolition (QND) measurement and feedback control. The manipulation of photons inevitably leads to nonlinear dynamics in the stochastic process. Here, the idea of robust control theory is applied to obtain a simple protocol for entanglement generation. We first analyze the generation of a single photon. A single-photon source has been developed in many different ways such as quantum dots [5], trapped single atom in a cavity [6], strong coupling between atoms and a cavity mode [7], parametric down-conversion [8], and conditional measurement on a cavity [9]. In contrast to these previous methods, we show a systematic design procedure using stabilization theory. Then we present feedback control for entangled photon generation by effectively realizing simultaneous measurements of total photon number and phase difference.

It is very important to note that our control strategy strongly depends on the robustness of feedback systems. It is well known that if there are no uncertainties in a system, feedback and feedforward are equivalent. The reason for using feedback is that feedback has a significant advantage in improving robustness to uncertainties. In our treatment, the stochastic noise in the system under QND measurement is regarded as uncertainties and feedback stabilizes the system against them.

The measurement model that we consider here consists of target and probe systems. The target systems contain observables to be measured, and the probe systems are used for the measurement of the observables. The experimental setup of measurement is depicted in Fig. 1. Denoted by $a_i$, $\alpha_i (i = 1, 2)$ are the mode operators of the target and probe systems with damping rates $k$ and $\kappa$, respectively. The target and probe systems interact with each other through a Hamiltonian $M_i = Q_i \chi \alpha_i^\dagger \alpha_i$, where $Q_i$ is the observable of the target systems and $\chi$ is the interaction coefficient. In the case of a cross-Kerr interaction, for example, $Q_i = a_i^\dagger a_i \equiv n_i$ is the number operator of each target system. Let $H_i$ be a control Hamiltonian in the target systems and $H \equiv \sum_i (H_i + M_i)$. The output of the second system is fed back to the first one through a beam splitter whose reflectivity and transmissivity are $\theta$ and $\phi$, respectively. If $\theta = 0$, the system becomes the cascade of the two

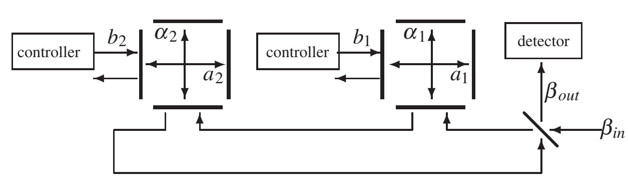


FIG. 1. The experimental setup. The probe inputs and outputs are split by optical isolators. The reflectivity and transmissivity of the diagonal beam splitter is $\theta$ and $\phi$, respectively. Each system has the cavity pumping control input along $b_i$.

 

cavities [10]. In the general case, the unitary operator of the whole system for a time interval $[t, t+dt]$ is given by [11]

$$U = \exp\bigg[iHdt + \sqrt{k}\sum_i (a_i^\dagger db_i - db_i^\dagger a_i) + \frac{\phi}{1+\theta}\sqrt{\kappa}\bigg\{\bigg(\sum_i \alpha_i\bigg)^\dagger d\beta_{\rm in} - d\beta_{\rm in}^\dagger\bigg(\sum_i \alpha_i\bigg)\bigg\} + \frac{\kappa}{2}(\alpha_2^\dagger \alpha_1 - \alpha_1\alpha_2^\dagger)dt\bigg], \quad (1)$$

where $b_i (i = 1, 2)$ are independent quantum Brownian noise to the target systems and $\beta_{\rm in}$ is incident light to the probe systems. We assume that $\beta_{\rm in} = \epsilon + i\lambda\sqrt{\kappa}$, where $\epsilon$ is another quantum Brownian noise independent of $b_i$ and $i\lambda\sqrt{\kappa}$ is a constant driving field.

Let us assume that $\kappa \gg \chi\langle Q_i\rangle$ and adiabatically eliminate the probe modes so that the unitary operator is expressed by the operators of the target systems alone. In this case, each probe mode can be approximated as

$$\alpha_1 = -\frac{2}{\sqrt{\kappa}}\bigg[1 + \frac{2}{\kappa}i\chi\bigg(\frac{1-\theta}{1+\theta}Q - Q_2\bigg)\bigg]\frac{\phi}{1+\theta}\beta_{\rm in}, \quad (2a)$$

$$\alpha_2 = \frac{2}{\sqrt{\kappa}}\bigg[1 + \frac{2}{\kappa}i\chi\bigg(\frac{1-\theta}{1+\theta}Q + Q_1\bigg)\bigg]\frac{\phi}{1+\theta}\beta_{\rm in}, \quad (2b)$$

where $Q = \sum_i Q_i$. Substituting this equation into (1) and letting $c = [8\lambda\chi(1-\theta)]/[\sqrt{\kappa}(1+\theta)]$ yield

$$U = \exp\bigg[i\bigg(4\lambda^2\chi Q + \sum_i H_i\bigg)dt + \sqrt{k}\sum_i(a_i^\dagger db_i - db_i^\dagger a_i) + \frac{c}{2}Q(d\epsilon - d\epsilon^\dagger)\bigg]. \quad (3)$$

The dynamics of the system are given by $X(t+dt) = UX(t)U^\dagger$ for any operator $X$, and the measurement process is the homodyne photon current of the output $\beta_{\rm out}$. These two processes constitute a *state equation*:

$$dX = \mathcal{L}Xdt + \frac{c}{2}[Q, X](d\epsilon - d\epsilon^\dagger) + \sqrt{k}\sum_i([a_i^\dagger, X]db_i + [X, a_i]db_i^\dagger), \quad (4a)$$

$$dz = -cQdt + d\beta_{\rm in} + d\beta_{\rm in}^\dagger, \quad (4b)$$

where $z = \beta_{\rm out} + \beta_{\rm out}^\dagger$ and

$$\mathcal{L}X = i\bigg[4\lambda^2\chi Q + \sum_i H_i, X\bigg] + \frac{c^2}{8}(2QXQ - Q^2X - XQ^2) + \sum_i k\bigg(a_i^\dagger X a_i - \frac{1}{2}(a_i^\dagger a_i X + X a_i^\dagger a_i)\bigg). \quad (5)$$

The system under measurement is described by the conditional process of the state equation (4). For any system operator $X$, a conditional expectation $\langle X\rangle$ is given by

$$d\langle X\rangle = \langle \mathcal{L}X\rangle dt + \bigg(\frac{c}{2}\langle QX + XQ\rangle - c\langle Q\rangle\langle X\rangle\bigg)dw, \quad (6)$$

where $dw = dz - c\langle Q\rangle dt$ is classical Brownian noise. This equation is used to control the system under QND measurement.

The advantage of using QND measurement for control is that the measured observable becomes deterministic asymptotically. Since the control input is represented by a Hamiltonian on the system Hilbert space, it cannot change the entropy of the system. The reduction in the entropy is thus determined only by the amount of information extracted from the system via measurement. In the case of QND measurement with unit efficiency, one can obtain perfect information of the measured observable asymptotically and the system results in a space spanned by eigenstates corresponding to a measurement outcome. Since for $X = Q$, the strength of the stochastic term of (6) is determined by the variance $\langle Q^2\rangle - \langle Q\rangle^2$, the stochastic noise to $\langle Q\rangle$ is attenuated over time under QND measurement. Thus, regarding the stochastic term of (6) as uncertainties in the system and utilizing the robustness of feedback control, one can expect that the system under QND measurement is controlled by designing an input only for the deterministic terms of (6).

Let us first apply this idea to photon number control of a single target system, $Q_1 = n_1$, $Q_2 = 0$, with a cavity pumping input along $b_1$. Assume that the damping rate is so small $k \sim 0$ that the pumping control can be described by a Hamiltonian $H_1 = u_1 y_1/2$, where $u_1$ is a control input and quadrature operators are defined as $x_1 = a_1 + a_1^\dagger$, $y_1 = -i(a_1 - a_1^\dagger)$. Note that the first term of the Hamiltonian in (5), $i4\lambda^2\chi n_1$, can be ignored since it represents a harmonic oscillation and the modification of control can be easily obtained. Then, the deterministic part of the system (6) is expressed as

$$\langle \dot{x}_1\rangle = -\frac{c^2}{8}\langle x_1\rangle + u_1, \quad (7a)$$

$$\langle \dot{n}_1\rangle = \frac{u_1}{2}\langle x_1\rangle. \quad (7b)$$

Our purpose is to construct the input $u_1$ to drive the system to a specific eigenstate $|\bar{n}_1\rangle$ of the number operator $n_1$. From the property of QND measurement above, this can be done by stabilizing the deterministic part (7) at $(\langle x_1\rangle, \langle n_1\rangle) = (0, \bar{n}_1)$ by feedback since $\langle x_1\rangle = 0$ and $\langle n_1\rangle = \bar{n}_1$ if the system is in $|\bar{n}_1\rangle$.

We first notice that no control can deterministically reduce the photon number since it follows from (7) that

$$\frac{d}{dt}\bigg(\langle n_1\rangle - \frac{\langle x_1\rangle^2}{4}\bigg) = \frac{c^2}{8}\langle x_1\rangle^2. \quad (8)$$

In other words, for any $t \geq 0$, the system is confined to

$$\langle n_1(t)\rangle \geq \frac{\langle x_1(t)\rangle^2}{4} + C, \quad (9)$$

where $C = \langle n_1(0)\rangle - \langle x_1(0)\rangle^2/4$. Hence if the system is initially in a state with $\langle x_1(0)\rangle = 0$, as satisfied by a standard initial state such as a vacuum or thermal equilibrium, then the photon number can only increase from the initial value, i.e., $\bar{n}_1 \geq \langle n_1(0)\rangle$.

To find a desirable controller under the constraint above, consider the form $\dot{u}_1 = f(\langle x_1\rangle, \langle n_1\rangle, u_1)$, where $f$ is a function to be designed. Dynamics introduced in the controller allows greater flexibility for control design and performance. Instead, $f$ is confined to a function which stabilizes $u_1$ itself. Hence $f$ is to be designed to stabilize the system at $(\langle x_1\rangle, \langle n_1\rangle, u_1) = (0, \bar{n}_1, 0)$. This can be achieved by a function

$$f = -ru_1 - p\langle x_1\rangle - q\langle x_1\rangle(\langle n_1\rangle - \bar{n}_1), \tag{10}$$

in which $p$, $q$, and $r$ are positive constants. The stability can be seen by taking a Lyapunov function $L = p\langle x_1\rangle^2 + 2q(\langle n_1\rangle - \bar{n}_1)^2 + u_1^2$. The asymptotic stability of this system can also be shown by sum of square [12]. A numerical result of this control is shown in Fig. 2(a), in which the system is initially prepared in a thermal state.

Let us analyze the behavior of this feedback system. It is generally desirable to produce the number state $|\bar{n}_1\rangle$ in the shortest possible time. However, since the reduction in the variance of $n_1$ cannot be changed by control as stated earlier, each parameter of the controller (10) is determined to generate high gain control to drive $\langle n_1\rangle$ to $\bar{n}_1$ before the variance decreases to zero. In this case, the behavior of the system can be described in two stages as shown in Fig. 3. Early in the control process, the controller generates a large input to drive $\langle n_1\rangle$ to $\bar{n}_1$ as soon as possible. Then, the first term of (7a) can be ignored and the system is confined on a quadratic curve

$$\langle n_1\rangle = \frac{\langle x_1\rangle^2}{4} + C. \tag{11}$$

As a result, $\langle n_1\rangle$ approaches $\bar{n}_1$ along this curve. In the next stage, since $\langle n_1\rangle \sim \bar{n}_1$ now, the control input becomes weaker so that the second term of (7a) can be ignored and the quadrature $\langle x_1\rangle$ converges to zero exponentially along a line $\langle n_1\rangle = \bar{n}_1$ due to the photon number-squeezing effect of QND measurement. At the end of the control process, the achieved number state is kept by constant pumping controls.

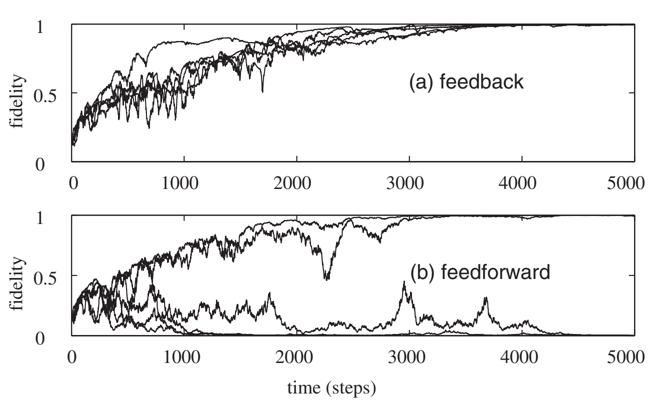


FIG. 2. Fidelity to a number state $|1\rangle_1$ subject to (a) the feedback control (10) and (b) feedforward control. The initial state is a thermal state with $\langle n_1(0)\rangle = 0.2$ for both cases.

From the consideration above, it is actually sufficient to stabilize only $\langle n_1\rangle$ along the curve (11) for producing a number state using QND measurement. In this case, control design is remarkably simplified. An example is given by a static controller $u_1 = q(\langle n_1\rangle - \bar{n}_1)$, where $q$ needs not be positive. Assume that $\langle x_1(0)\rangle = 0$. Then $q\langle x_1\rangle < 0$ since $\bar{n}_1 \geq \langle n_1\rangle$, and the stability of $\langle n_1\rangle$ can be seen by taking a Lyapunov function $L = (\langle n_1\rangle - \bar{n}_1)^2$.

Note that the deterministic part (7) is formally equivalent to the unconditional evolution of the system under measurement. One would expect that the system could be controlled by applying the same control philosophy to the nonconditional process, which leads to feedforward control. However, as shown in Fig. 2(b), feedforward control is generally sensitive to stochastic uncertainties and fails to produce a number state deterministically even though the stochastic noise is attenuated over time.

Now let us consider a case where the two target systems are subject to QND measurement. QND measurement has the potential to produce entanglement between the two systems. Note that as formulated in (5), the two systems are controlled independently. However, it is permitted to feed the same estimates back to both systems because the estimates are classical quantities.

Since there are many degenerate eigenstates of $Q_1 + Q_2$, a simple extension of number control does not work for entanglement generation. In general, this problem can be avoided by measuring an auxiliary observable commutative with $Q_1 + Q_2$, which is the phase difference operator in this case. Unfortunately, the present measurement setting in Fig. 1 does not have such a feature. We show that the phase difference can also be controlled by feedback.

Assume that the system is initially in a vacuum state and the control Hamiltonian for each target system is of the form $H_i = u_i y_i/2$ $(i = 1, 2)$ as before. From (6), the deterministic part of the system is given by

$$\langle \dot{x}_i\rangle = -\frac{c^2}{8}\langle x_i\rangle + u_i, \qquad \langle \dot{n}_i\rangle = \frac{u_i}{2}\langle x_i\rangle. \tag{12}$$

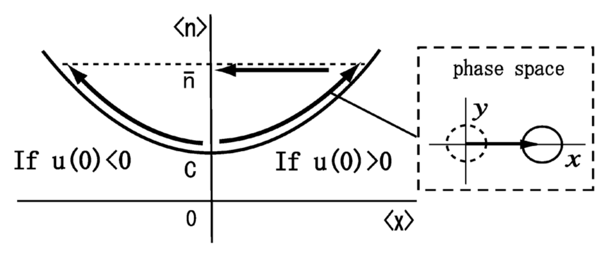


FIG. 3. The trajectory of the system. Because of (9), the system is confined to the upper side of the curve (11). In the first stage of control, the system approaches $\langle n_1\rangle = \bar{n}_1$ along the curve. Then, $\langle x_1\rangle$ is stabilized due to the squeezing effect of QND measurement. The sign of the initial input value determines which side the system goes, $\langle x_1\rangle > 0$ or $\langle x_1\rangle < 0$.

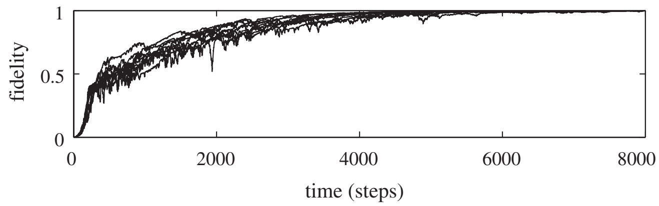


FIG. 4. Fidelity to $|1-\rangle$. The signs of the initial control inputs $u_i(0)$ ($i = 1, 2$) defined in (13) are taken to be different.

For the purpose of entanglement generation, we stabilize the system at not only $(\langle x_i\rangle, u_i) = (0, 0)$ but also $(\langle n_1 + n_2\rangle, \langle n_1 - n_2\rangle) = (1, 0)$. The same analysis as the single cavity case yields desirable control inputs:

$$\dot{u}_i = -r_i u_i - p_i\langle x_i\rangle - q\langle x_i\rangle(\langle n_1 + n_2\rangle - N) + (-1)^i s\langle x_i\rangle\langle n_1 - n_2\rangle \quad (i = 1, 2), \tag{13}$$

where $p_i$, $q$, $r_i$, and $s$ are positive constants and $N = 1$. As a result of this feedback control, there are two possible eigenstates to be obtained: $|1\pm\rangle \equiv (|01\rangle \pm |10\rangle)/\sqrt{2}$. In the control process these two states are distinguishable by how the system is stabilized. If the two systems approach the equilibrium point from the different side in Fig. 3, the system results in $|1-\rangle$. Since the sign of $\langle x_i\rangle$ is determined by that of the initial input value $u_i(0)$, $|1-\rangle$ is obtained by choosing different signs for $u_1(0)$ and $u_2(0)$, as shown in Fig. 4. Conversely, $|1+\rangle$ is obtained if we choose the same signs, i.e., $u_1(0)u_2(0) > 0$.

This can be explained in a simple way based on Fig. 3. Suppose that $u_1(0) > 0$ and $u_2(0) < 0$. Then, as stated above, $\langle x_1(t)\rangle > 0$ and $\langle x_2(t)\rangle < 0$ for $t > 0$. Immediately after the control is applied, the number-squeezing effect of the measurement does not start yet, so the state of the system can be approximated to $|\gamma\rangle_1 \otimes |-\gamma\rangle_2$, where $|\gamma\rangle$ is a coherent state with $1 \gg \gamma > 0$. In the number basis, this state is expressed as

$$|\gamma\rangle_1 \otimes |-\gamma\rangle_2 = e^{-\gamma^2}[|00\rangle - \gamma(|01\rangle - |01\rangle) + \gamma^2(|02\rangle - \sqrt{2}|11\rangle + |20\rangle)/\sqrt{2} + \cdots]. \tag{14}$$

The feedback control (13) amplifies the second term of (14) and enables us to obtain $|1-\rangle$ with probability 1. When we choose the same sign for $u_1(0)$ and $u_2(0)$, $|1+\rangle$ is obtained because the state of the system is approximated to $|\gamma\rangle \otimes |\gamma\rangle$ early in the control process. This control method can be used to produce a $W$-state type of entanglement for three or more systems. For example, if three systems are subject to QND measurement, we can obtain $|001\rangle \pm |010\rangle \pm |100\rangle$. It is now obvious how the phase of each term is selected.

If we stabilize the system at $(\langle n_1 + n_2\rangle, \langle n_1 - n_2\rangle) = (N, 0)$ for $N > 1$ using the controller (13) with $u_1(0) > 0$ and $u_2(0) < 0$, the $(N + 1)$th term of (14) is amplified with probability 1. The achieved fidelity to a maximally entangled state $|N-\rangle \equiv \sum_{k=0}^{N}(-1)^k|k, N - k\rangle/\sqrt{N + 1}$ is

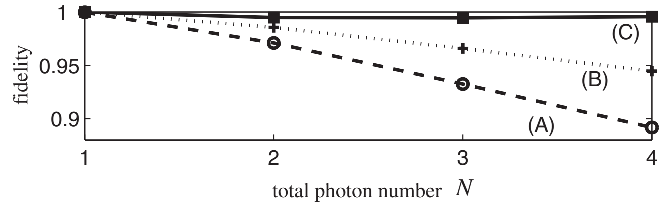


FIG. 5. Fidelity to $|N-\rangle$ attained by feedback (13) when the system is initially in a vacuum state [dashed line ($A$)] and in independently squeezed states [solid line ($C$)]. The initial squeezing is $\Delta y_i = e^{-0.5}$, and the achieved fidelity is 0.995 for $N = 2, 3, 4$. The dotted line ($B$) is the fidelity of the $(N - 1)$th term of (14) to $|N-\rangle$.

shown in Fig. 5 ($A$). The fidelity gets worse for higher order. This can be overcome by introducing an appropriate quadrature squeezing to the initial state. Suppose that the two systems are initially squeezed in the $y$ direction independently. Since they are driven along $x$ axis of the phase space in the first stage as in Fig. 3, the control inputs produce a state with small fluctuations in the phase difference between the two systems as if under phase difference measurement. Then, the number-squeezing effect of QND measurement starts in the second stage. As a result, we obtain the maximally entangled states of higher order deterministically, as shown in Fig. 5 ($C$).

We conclude from these examples that the combination of QND measurement and feedback control effectively realizes simultaneous measurements of total photon number and phase difference and control design is drastically simplified due to the robustness of feedback control.

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